ESD-TDR-64-172

TM-04026

BAR-MULTIPLIER AND ITS GENERALIZATION

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR-64-172

S. Okada

December 1965

Prepared for

DIRECTORATE OF COMPUTERS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE

L. G. Hanscom Field, Bedford, Massachusetts

Project 707.0

Prepared by

THE MITRE CORPORATION
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ABSTRACT

A new and simple device, similar to an abacus, was invented by a Chinese mathematician; it is used for multiplication, division, and extracting square roots. The underlying idea is abstracted and generalized for possible application to contemporary computers in order to reduce computing time.

REVIEW AND APPROVAL

This technical documentary report has been reviewed and is approved.

James A. Mellin Maj

for SEYMOUR JEFFERY
Major, USAF
Chief, Computer Division
Directorate of Computers

ACKNOWLEDGEMENT

The author wishes to express his thanks to Mr. B. A. Forest and Mrs. M. E. Warford, Department D-74, The MITRE Corporation, for their production of a model of the bar-multiplier and for creative discussion which was a stimulus for this work. He also thanks Messrs. N. S. Zimbel and J. Terzian, and Mrs. J. W. S. Liu of Department D-71, The MITRE Corporation, for their constructive discussions during the preparation of the paper.

The author is further indebted to Mr. Terzian for suggesting the method for the solution of the binary case illustrated in Figure 11.

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SECTION I

INTRODUCTION

A very simple device was recently invented in China for effecting numerical multiplication, division, and extracting square roots, all without the use of the multiplication table.

According to a popular journal [Reference -cm] from which this information was obtained, the inventor, the mathematician YU Chan-Hsan, was originally a carpenter and later a clerk in a stationary store. A movie of his biography was produced because his invention is comparable only to that of the abacus. His bar-multiplier is to the process of multiplication what the abacus is to addition, and just as the abacus is universally used in China, YU's bar-multiplier in the combined form with the abacus is also becoming part of the standard tools of his country. Because it is so simple and easy to learn, it is now taught at almost all grammar schools of Mainland China.

"Napier's Bones," invented by John Napier (1550-1617), originator of numerical calculation by the logarithm, are basically identical with the method of Figure 1 [References -EB:cm, N-WM1, 461]. The new Chinese method differs from it in

		A =	1	4	2	3
	X	B =		8	3	2
			2	8	4	6
A_iB_j		3	12	6	9	
	+ 8	32	16	24		
Q_t	8	35	30	38	13	6
	3	5	3	8		6
	8	3	0	1	3	
P = 1	1	8	3	9	3	6

Figure 1 An Example of Multiplication

the use of the bars instead of the numerals and in combining the bar-multiplier with the abacus as the "bar-bead calculator." It converts the arithmetic operations into discrete, purely geometric operations on points and lines such as embodying lines, by determining intersections in the bar part and linear displacement of beads in the abacus part.

This document explains YU's bar-calculator as a prototype of the new idea, and then generalizes it to the multiplication of many numbers so the numerical computation of the N-th root becomes simple and fast.

Application of the basic principle of the prototype gives a possibility of reducing computing time with some increase in computing elements. For the generalized case, further reduction is possible by a further increase of the computing elements. The key lies in the separation of the pure multiplying operation of individual digits from the carrying-over process, which can be performed in the subsequent adding step.

When an equation is valid only for the multiplication of N = 2 or 3 digits, namely AB or ABC, it is shown by adding the denominator N to the number of the equation which is valid for the general value N. Variables are indicated by lower-case letters; constants are indicated by capital letters. All lower-case letter indices are variable and all upper-case letter indices are fixed.

From the standpoint of modern abstract algebra, if the ordinary carried-over product P is substituted by the "quasi-decimal," "quasi-binary" or any "quasi" digital expression Q of the product defined below by withholding the carryover process, a lucid isomorphism between the digital numbers of any single base and the functions of a single variable is revealed.

As a whole, this paper means the realization of direct isomorphic mapping of the arithmetic operations on numbers into discrete affine geometric operations such as meet or join on points, lines, planes, or hyperplanes by means of the logical electric circuits.

Though some literature [C-GN, F-SZ, H-MGN, L-DG, M-GZ] on the "geometry of numbers" already exists, none includes the geometrical method of multiplying digital numbers by discrete models, which also seems to deserve the term "geometry of numbers." Closely related problems are discussed in computer design, but without geometric interpretation [R-ADC].

The generalization is easily obtained by considering the diagonal hyperplanes expressed by the covectors in the N-dimensional affine space. Though the following explanation is given mostly on the decimal case, the result is valid for any base, and would be particularly advantageous in the binary system.

SECTION II

PRINCIPLE OF THE BAR-MULTIPLIER

For the multiplication of two numbers:

$$n = A, \; B \qquad 1/2$$

$$P = AB \qquad 2/2$$

where

$$P = P_S 10^S + \ldots + P_s 10^s + \ldots + P_o \, , \; P_S \neq 0, \; P_s < 10 \, , \qquad 3$$

$$s = 0 \, , \; 1, \ldots , \; S ; \qquad 4$$

$$A = A_I 10^I + \ldots + A_i 10^i + \ldots + A_o \, , \; A_I \neq 0 \, , \; A_i < 10 \, , \qquad 5.1$$

$$i = 0 \, , \; 1 \, , \ldots , \; I ; \qquad 6.1$$

$$B = B_J 10^J + \ldots + B_j 10^j + \ldots + B_o \, , \; B_J \neq 0 \, , \; B_j < 10 \, , \qquad 5.2$$

$$j = 0 \, , \; 1 \, , \ldots , \; J ; \qquad 6.2$$

if the products $A_i \, B_j$ for all pairs of digits A_i and B_j are written down, deferring the carry-over from $A_i \, B_j$ to $A_{i+1} B_j$; e.g., 1 of $A_2 B_1 = 4.3 = 12$ of Figures 1 and 2 to $A_3 B_1 = 1.3 = 3$ into the final adding step, the multiplying process will change to the form shown in Figures 1 and 2.

In the binary system there is no carry-over in the partial product AB_j. In Figure 1, the two-digit number of the sum of each column are vertically staggered with underlines for the convenience of the succeeding carry-over step. Thus, the products of the type $A_i B_j$ are perfectly separated from each other, and are mutually independent in logic. By this separation, the geometrical location of the factors A_i and B_j in Figurc 2 becomes very simple: all B_o appear in the first row of the

$$
\begin{array}{ccccccc}
 & & & A_3 & A_2 & A_1 & A_o \\
 & \times & & & B_2 & B_1 & B_o \\
\hline
 & & & A_3B_o & A_2B_o & A_1B_o & A_oB_o \\
 & & A_3B_1 & A_2B_1 & A_1B_1 & A_oB_1 & \\
+ & A_3B_2 & A_2B_2 & A_1B_2 & A_oB_2 & & \\
\hline
 & Q_5 & Q_4 & Q_3 & Q_2 & Q_1 & Q_o \\
 & & & & & & \\
\hline
P = P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & P_o
\end{array}
$$

Figure 2 The General Case

uncarried partial product and all A_o appear in the first right skew line. Therefore, as shown in Figure 3, if two lines are drawn horizontally through the positions of all B_o and three skew lines are drawn through the positions of A_o, then the number of the cross-points of these lines at the intersection corresponding to A_oB_o is exactly $A_oB_o = 2.3 = 6$. Thus, if all numerals A_i and B_j are substituted by lines as shown in Figure 3, the number of the cross-points at all of

$$M_2 = (I + 1)\ (J + 1) = 4.3 = 12 \qquad 7/2$$

intersections gives the values of the products A_iB_j.

If both are D digits:

$$I + 1 = J + 1 = D\ , \qquad 8/2$$

then,

$$M_2 = D^2\ . \qquad 9/2$$

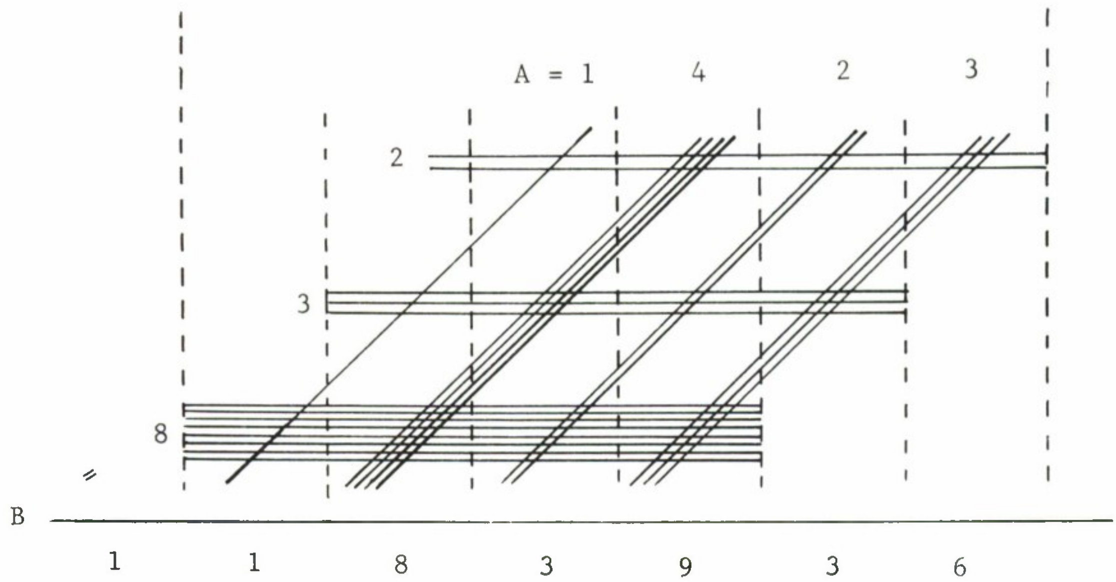
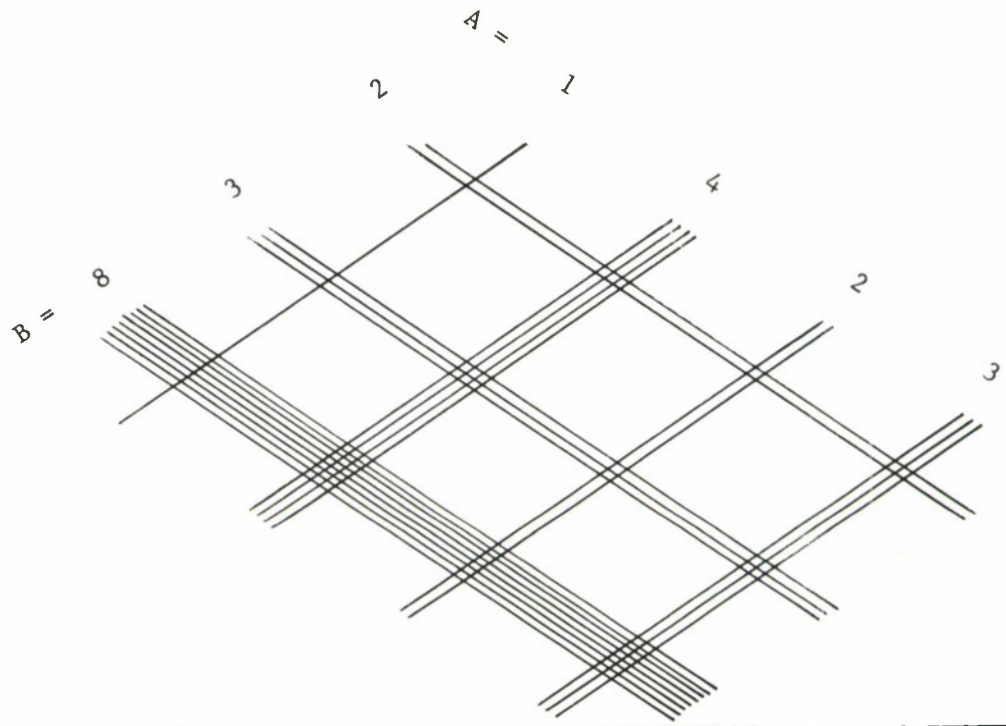


Figure 3 Bar-Multiplier

Because A and B are commutative, by inclining the lines of B_j as shown in Figure 4, A and B are brought into symmetry. If the bar-multiplier is combined with a Japanese abacus (which has only five beads for each digit—one upper bead expressing five and four lower beads expressing unity), the number of bars for each digit can be similarly reduced to five.

Further, instead of removable bars, flat bars can be installed which can be turned a certain angle by small staggered levers so that each digit can be expressed by an angular flipping of these bars. Now based on Figure 4, the principle of the "bar-multiplier" is stated as follows:

If two numbers A and B, e.g., 1423 and 832, are to be multiplied, place or draw one, four, two and three parallel inclined bars from the left lower position to the upper right, and eight, three, and two inclined bars from the left upper position to the right lower as shown in Figure 4 then add the number of the cross-points along the vertical columns with carry over, to obtain the product P, e.g., 1183936.



	8	3	0	1	3	6
Q_t	3	5	3	8		
P = 1	1	8	3	9	3	6

Figure 4 Bar-Multiplier in Symmetric Form

If $P = 1183936$ and $A = 1423$ are given, B can be determined by subtracting the cross-points generated by the maximum digits B_j. Thus division is possible as exemplified for $68373 \div 273$ in Figure 5. Similarly, square and square root can be performed by this bar-multiplier as exemplified for 213^2 in Figure 6 and for $45369^{1/2}$ in Figure 7.

2P PROOF

In the products $A_i B_j 10^{i+j}$, at first the sum $Q_t 10^t$ of all the terms of power t (see Figures 1 and 2) is defined by

$$Q_t = \Sigma_{i,j} R_{ij} , \qquad 10/2$$

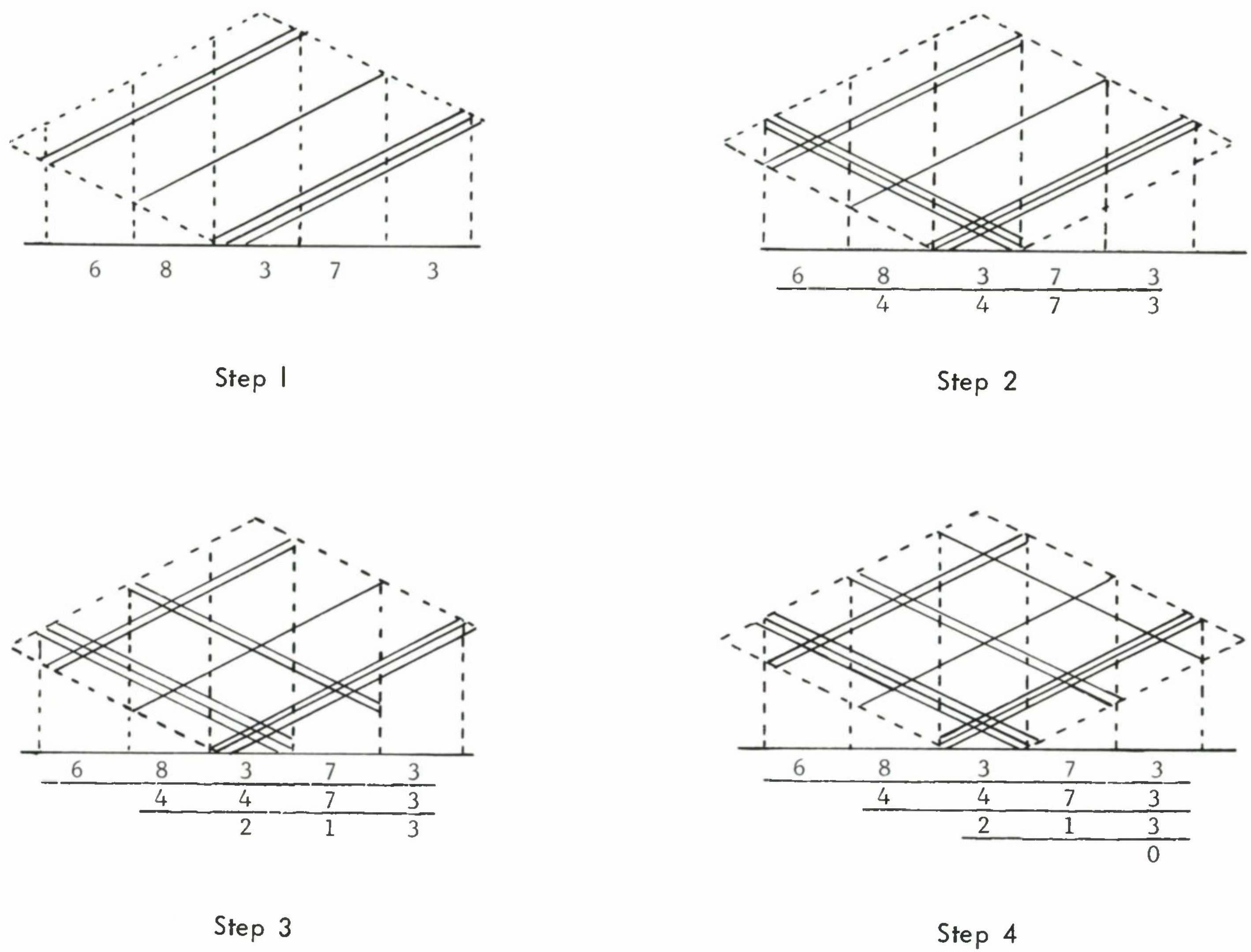


Figure 5 Dividing 68373 by 213

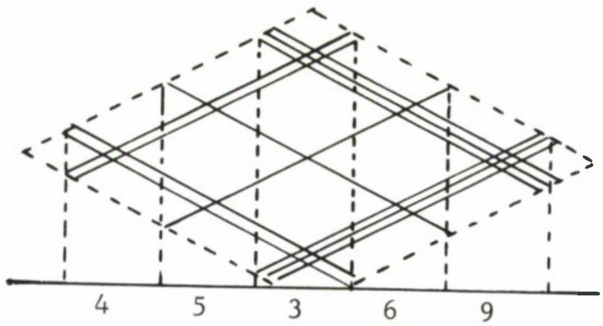


Figure 6 Squaring 213

$$R_{ij} = A_i B_j \qquad 11/2$$

over all values of i and j of

6.1, 6.2: $\quad i = 0, 1, \ldots, I; \; j = 0, 1, \ldots, J;$

satisfying

$$i + j = t, \qquad 12/2$$

$$t = 0, 1, \ldots, T; \qquad 13$$

$$T = I + J. \qquad 14/2$$

Q is defined as the sum of all $Q_t 10^t$:

$$Q = Q_T 10^T + \ldots + Q_t 10^t + \ldots + Q_0. \qquad 15$$

Though Q is equal to the product P:

$$Q = P, \qquad 16$$

Q_t may generally be not only larger than ten:

$$Q_t \lesseqgtr 10 \qquad 17$$

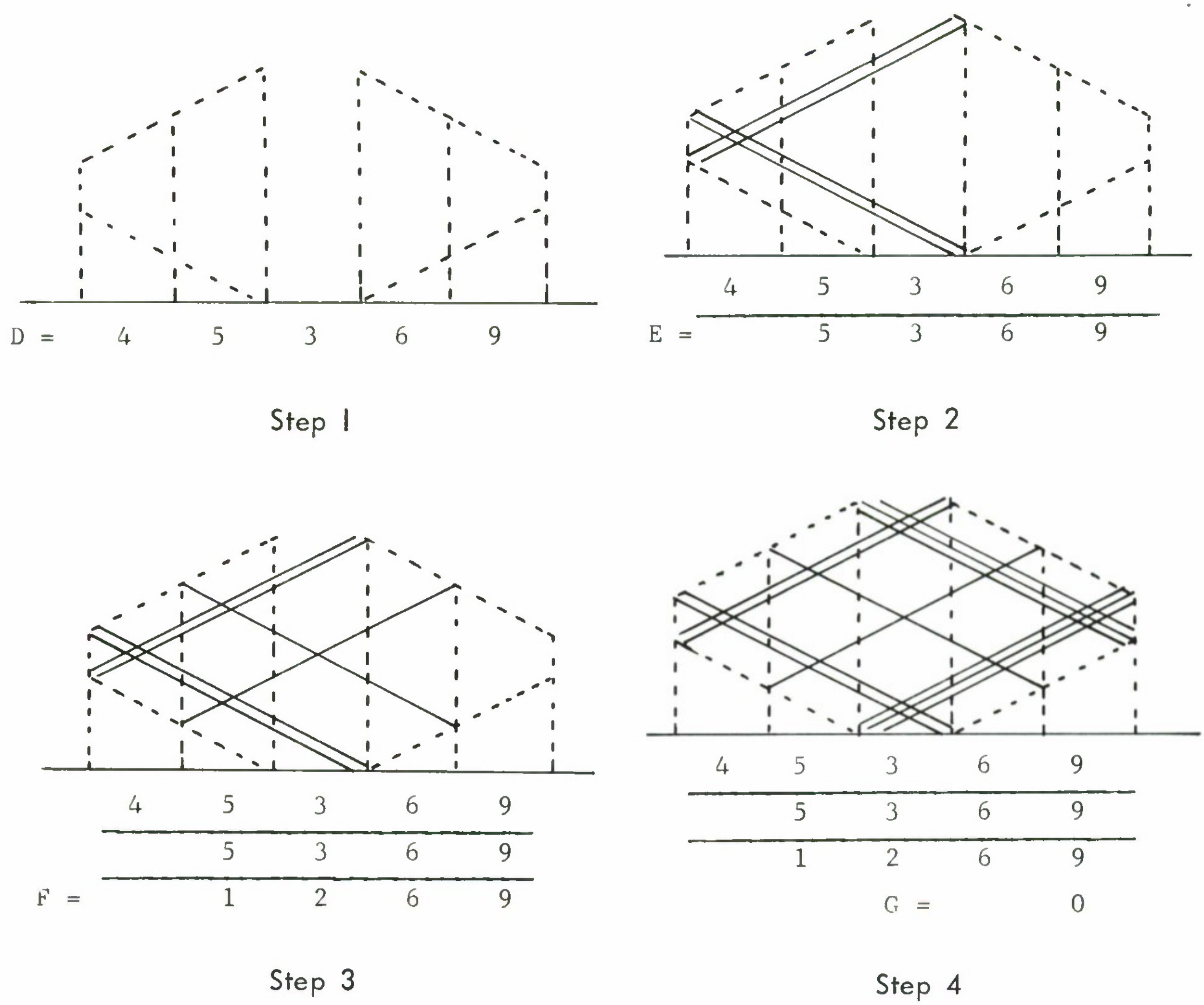


Figure 7 Square Root of 45369

as seen in the example of Figure 1, but it may consist of three or more digits. Hence, Q will be ealled the "quasi-decimal" expression of the produet. Now, on the plane of the paper of Figure 4, if one introduees an affine frame (see the end of this paragraph) having the bases

$$\vec{a}_1 = \overrightarrow{0A}_1, \; \vec{a}_2 = \overrightarrow{0A}_2 \qquad \text{a-1/2}$$

where the origin 0 is at the point R_{00}, the first base point A_1 is at the point R_{10}, the second base point A_2 is at the point R_{01} (as shown in Figure 8), then a linear equation

$$w_1 x^1 + w_2 x^2 = 1 \qquad \text{a-2/2}$$

on the components of a vector $\vec{x}$ where

$$\vec{x} = x^1 \vec{a}_1 + x^2 \vec{a}_2 \qquad \text{a-3/2}$$

expresses a straight line intersecting with 1-axis at $1/w_1$ and 2-axis at $1/w_2$ and

$$w_1 x^1 + w_2 x^2 = t;\ t = 0, 1, \ldots, T \qquad \text{a-4/2}$$

express straight line in parallel to that of equation a-2/2 with the intercepts t/w_1 and t/w_2 on 1-axis and 2-axis. w_1 and w_2 are covariant referring to x^n (n = 1, 2) and called the components of a covector $\underset{\rightarrow}{w}$:

$$\underset{\rightarrow}{w} = w_1 \underset{\rightarrow}{a}^1 + w_2 \underset{\rightarrow}{a}^2 \qquad \text{a-5/2}$$

where

$$\underset{\rightarrow}{a}^1 = \underset{\rightarrow}{0A}^1 \qquad \text{a-6.1/2}$$

is the base covector expressed by a pair of an initial straight line 0 passing through the origin and a terminal straight line A^2 passing through the base point A_1, both in parallel to 2-axis.

Thus,

$$\underset{\rightarrow}{a}^2 = \underset{\rightarrow}{0A}^2 \qquad \text{a-6.2/2}$$

is the base covector expressed by a pair of an initial straight line 0 passing through the origin and a terminal straight line A^2 passing through the base point A_2, both in parallel to 1-axis. These base covectors $\underset{\rightarrow}{a}^1$ and $\underset{\rightarrow}{a}^2$ are contravariant as expressed by their superscript. If the components of the covector $\underset{\rightarrow}{w}$ are both unity,

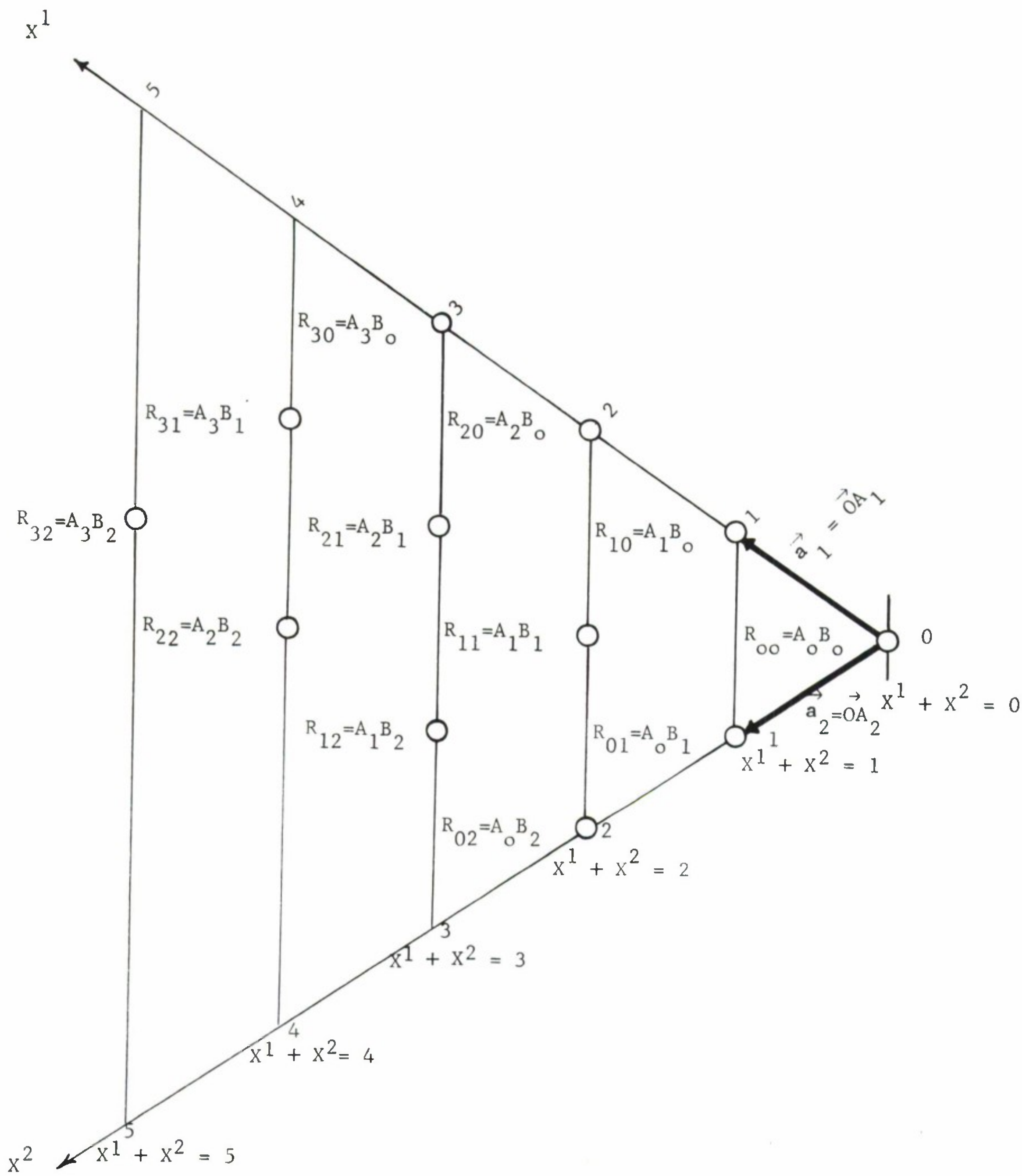


Figure 8 Affine Coordinates for Figure 4

$$(w_1, w_2) = (1, 1) \ , \qquad \text{a-7/2}$$

the corresponding equation

$$x^1 + x^2 = 1 \qquad \text{a-8/2}$$

expresses the straight line passing the base points A_1 and A_2; namely, a part, A_1A_2, of it forms the diagonal of the base parallelotope. Similarly,

$$x^1 + x^2 = t \ , \ t = 0, 1, \ldots , T \qquad \text{a-9/2}$$

expresses the straight line passing the points (t, 0) and (0, t) in parallel to that of equation a-8/2.

By comparison of equations 12/2, 13, and a-9/2, we see that Q_t is the sum of the products R_{ij} lying on the diagonal lines of equation a-9/2 as seen in Figures 4 and 8.

Thus the validity of the above principle was proved by introducing the affine frame. If this affine frame is used, the above process can be stated as follows as the case 2 of the general case N described below:

1. Construct a two-dimensional affine frame;

2. Enter A_i to all points

$$x^2 = j \ , \ j = 0, \ldots , J \qquad 18.1/2$$

on the line

$$x^1 = i \ , \ i = 0, \ldots , I \ ; \qquad 19.1/2$$

3. Enter B_j to all points:

$$x^1 = i \ , \ i = 0, \ldots , I \qquad 18.2/2$$

on the line

$$x^2 = j\ ,\ j = 0, \ldots, J\ ; \qquad 19.2/2$$

4. By adding all products R_{ij} on the diagonal lines

a-9/2: $x^1 + x^2 = t$, 13: $t = 0, \ldots, T$; 14/2: $T = I + J$ 20/2

Q_t and hence Q are obtained;

5. By carrying, P is obtained from Q.

A note on the "affine" space [see W-STM, S-RC 1, Wh-GI 349-53, G-LA 250, A-GA 66]: If different lengths are used for units of different axes of an oblique coordinate system in N-dimensional Euclidean space, namely a "heterometric" oblique frame is used, the frame can be called the "affine" frame without loss of generality. The unit of length of any direction other than those of the axes is not defined and is perfectly arbitrary. The inner (or dot) product is defined only for a vector $\vec{v} = \overrightarrow{0V}$ and a covector $\underset{\rightarrow}{w} = \underset{\rightarrow}{0W}$ where $\underset{\rightarrow}{0W}$ means a pair of initial and terminal hyperplanes in parallel, with intercepts $1/w_n$ on n-axis ($n = 1, 2, \ldots, N$) (straight lines for $N = 2$, planes for $N = 3$). When the origin is fixed, the initial hyperplane passing through the origin is often omitted. The value of the inner product expresses the ratio of the length 0V to the intercept 0W of the vector $\vec{v} = \overrightarrow{0V}$ by the covector $\underset{\rightarrow}{w}$, where W means the intersection of 0V or its extension with the terminal hyperplane of the covector $\underset{\rightarrow}{w}$ [S-RC8-9]:

$$\vec{v}\,\underset{\rightarrow}{w} = v^1 w_1 + \ldots + v^N w_N = v^n w_n = 0V:0W\ . \qquad \text{a-10}$$

When the vector $\vec{v}$ is in parallel with the hyperplane, namely, both 0 and V are in the initial hyperplane, this product becomes zero, which is the case of $t = 0$ in equation a-4/2. When the intersecting point W is on the extension of $\overrightarrow{V0} = -\vec{v}$ beyond the origin 0, the product becomes negative.

Hence, the equation

$$w_n x^n = w_1 x^1 + \ldots + w_N x^N = 1 \qquad \text{a-2}$$

expresses a hyperplane passing the point $1/w_n$ of the n-th axis, and x^n satisfying

$$w_n x^n = t \; ; \; t = 0, 1, \ldots, T \qquad \text{a-4}$$

expresses a hyperplane passing through the point t/w_n of n-th axis, in parallel to that of a-2. If all of w_n were unity:

$$w_n = 1, \quad n = 1, 2, \ldots, N, \qquad \text{a-7}$$

the resulting equation

$$x^1 + x^2 + \ldots + x^N = 1 \qquad \text{a-8}$$

expresses a diagonal hyperplane passing through all base points $A_1, A_2, \ldots, A_N$ and

$$x^1 + x^2 + \ldots + x^N = t, \quad t = 0, 1, \ldots, T \qquad \text{a-9}$$

expresses diagonal hyperplanes intersecting with n-axis at the length t hence in parallel to that of a-8.

If the base vectors are expressed as

$$\vec{a}_n = \overrightarrow{0A}_n, \quad n = 1, 2, \ldots, N, \qquad \text{a-1}$$

base covectors as

$$\underset{\rightarrow}{a}^n = \underset{\rightarrow}{0A}^n, \qquad \text{a-6}$$

then there holds

$$\vec{a}_n \underset{\rightarrow}{a}^m = 1_n^m, \quad m, n = 1, 2, \ldots, N \qquad \text{a-11}$$

where 1_n^m means the unit matrix if the left hand side $\vec{a}_n \underset{\rightarrow}{a}^m$ is regarded as a matrix. As an equation in the index notation, 1_n^m is identical with the Kronecker symbol. Unless an orthonormal frame is defined or a covector $\underset{\rightarrow}{w}$ is substituted by a vector $\vec{w}$ under a certain geometric correspondence, an "inner product between two vectors" and hence also an "angle" are not defined and the Pythagorean theorem is not applicable. If only the affine frames are used, the space is called the "affine space."

Since an "N-dimensional space" for any natural number N is none other than a set of points and substantially means that any combination of the numerical value of N variables is mapped to a single point in this set, any N-space and N-geometric object can be drawn on paper as the projected figure of the N-dimensional figure into the two-dimensional space in exactly the same way as practiced in the graphics and descriptive geometry. Especially, if the geometric figures are discrete, this projected realization in 2-space can be proper two-dimensional figures.

N-space ($N > 3$) is already often used among engineers [A-gemsf, H-gs 172, 183-98]. Physical realization of N-space by means of electric networks was explained in 1940 by L. Gutenmacher in G-kmuk [see also K-NG, S-MG-1, -2, S-NG, C-RP 141, etc.].

SECTION III

MULTIPLICATION OF THREE NUMBERS

The geometrical arrangement of this case is exemplified in Figure 9, but the carrying circuits in Figure 9 are for the binary case which is explained later. In case of

$$n = A,\ B,\ C \qquad 1/3$$

$$P = ABC \qquad 2/3$$

$$A = A_I 10^I + \ldots + A_i 10^i + \ldots + A_0,\ A_I \neq 0,\ A_i < 10 \qquad 5.1$$

$$B = B_J 10^J + \ldots + B_j 10^j + \ldots + B_0,\ B_J \neq 0,\ B_j < 10 \qquad 5.2$$

$$C = C_K 10^K + \ldots + C_k 10^k + \ldots + C_0,\ C_K \neq 0,\ C_k < 10\ . \qquad 5.3$$

The product P is obtained by the carrying-over process from the quasi-decimal expression Q defined by equations 13 and 15 and from the following equation 14/3 and the sum

$$Q_t = \Sigma_{i,\,j\,;k} R_{ijk} \qquad 10/3$$

$$t = 0,\ 1,\ \ldots,\ T \qquad 13$$

$$T = I + J + K \qquad 14/3$$

$$R_{ijk} = A_i B_j C_k \qquad 11/3$$

over all the values of i, j, and k satisfying

$$i + j + k = t \qquad 12/3$$

$$i = 0,\ 1,\ \ldots,\ I \qquad 6.1$$

$$j = 0,\ 1,\ \ldots,\ J \qquad 6.2$$

$$k = 0,\ 1,\ \ldots,\ K\ . \qquad 6.3$$

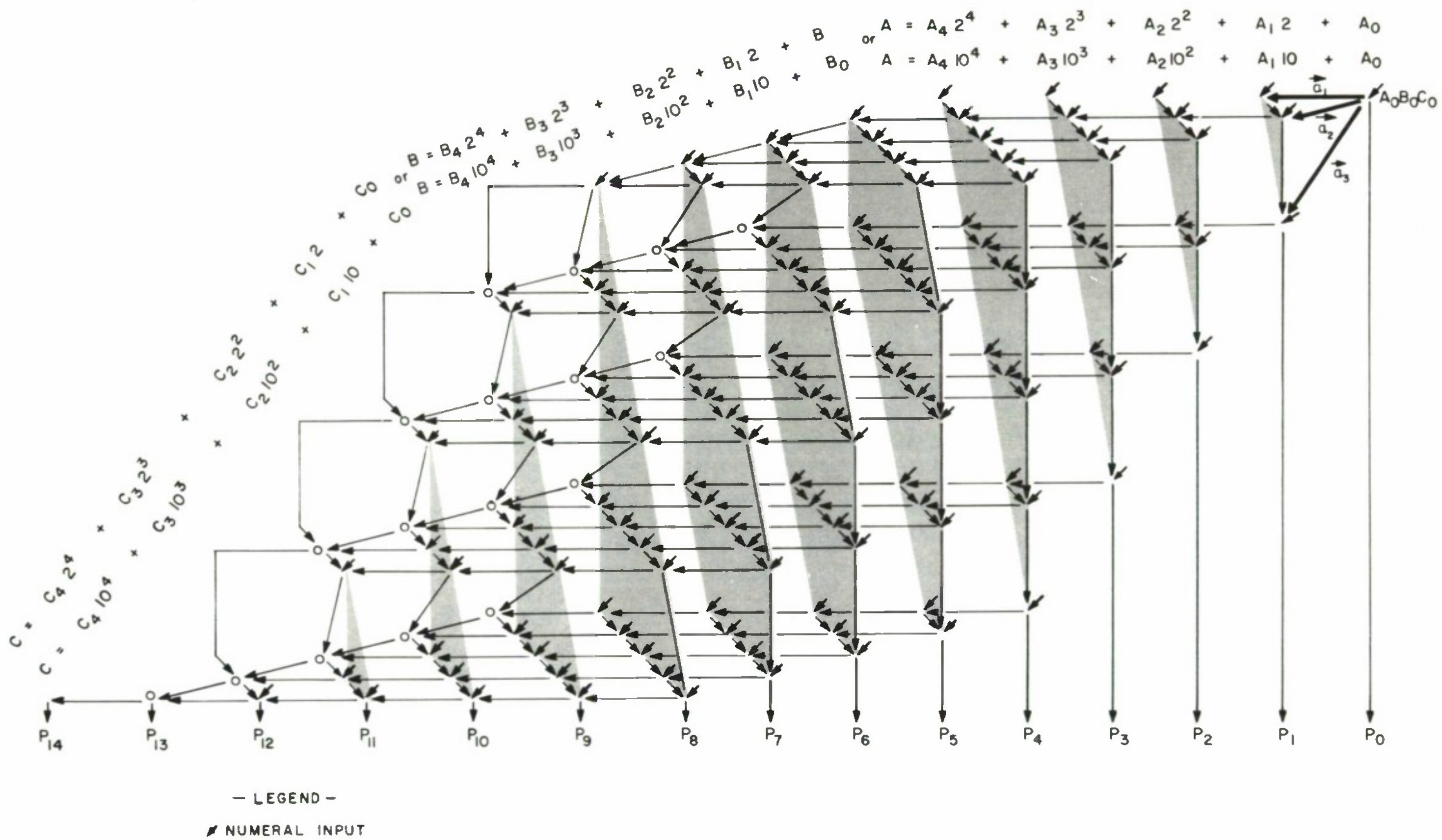


Figure 9. Multiplier of Three Numbers - A, B, and C

If any three-dimensional affine frame is constructed as shown in the three-dimensional Figure 9 and we

1. write down the value A_i at all points of the coordinates:

$$(x^2, x^3) = (j, k), \quad (j = 0, 1, \ldots, J ; k = 0, 1, \ldots, K) \qquad 18.\ 1/3$$

on the plane:

$$x^1 = i, \quad (i = 0, 1, \ldots, I) \qquad 19.\ 1/3$$

2. write down the value B_j at all points of the coordinates:

$$(x^1, x^3) = (i, k) \quad (i = 0, 1, \ldots, I ; k = 0, 1, \ldots, K) \qquad 18.\ 2/3$$

on the plane:

$$x^2 = j, \quad (j = 0, 1, \ldots, J) \qquad 19.\ 2/3$$

3. write down the value C_k at all points of the coordinates:

$$(x^1, x^2) = (i, j) \quad (i = 0, 1, \ldots, I ; j = 0, 1, \ldots, J) \qquad 18.\ 3/3$$

on the plane:

$$x^3 = k, \quad (k = 0, 1, \ldots, K) \qquad 19.\ 3/3$$

then

4. the summands of Q_t lie on the diagonal plane

a-9/3: $$x^1 + x^2 + x^3 = t \qquad 20/3$$

$$t = 0, 1, \ldots, T ; \qquad 20/3$$

$$T = I + J + K$$

From these, the procedures for multiplication are established as follows:

1. Place A_i sheets of planes at

$$x^1 = i, \ i = 0, 1, \dots, I \qquad 21.1/3$$

in parallel to the plane

$$x^1 = 0 \qquad 22.1/3$$

namely, the initial plane of a covector of components

$$(w_1, w_2, w_3) = (1, 0, 0) \ ; \qquad 23.1/3$$

2. Place B_j sheets at

$$x^2 = j, \ j = 0, 1, \dots, J \qquad 21.2/3$$

in parallel to the plane

$$x^2 = 0 \ ; \qquad 22.2/3$$

3. Place C_k sheets at

$$x^3 = k, \ k = 0, 1, \dots, K \qquad 21.3/3$$

in parallel to the plane

$$x^3 = 0 \ . \qquad 22.3/3$$

4. Add the number of intersection points of these sets of plane on the diagonal planes of equation 19/3 with carry from Q_t over to Q_{t+1} for $t < T$ and from Q_T to P_{T+1} or further to P_{T+2} etc. Then the product P is obtained. The total number of the product terms $A_iB_jC_k$ is given by

$$M_3 = (I + 1)(J + 1)(K + 1) \ . \qquad 7/3$$

If all three numbers are of D digits:

$$I + 1 = J + 1 = K + 1 = D \ , \tag{8/3}$$

then there holds

$$M_3 = D^3 \ . \tag{9/3}$$

SECTION IV

MULTIPLICATION OF N NUMBERS

In case

$$n = A, B, \ldots , N ;$$ 1

$$P = AB \ldots n \ldots N ,$$ 2

$$n = n_{I^n} 10^{I^n} + \ldots + n_{i^n} 10^{i^n} + \ldots + n_0, \; n_{I^n} \neq 0, \; n_{i^n} < 10$$ 5

$$i^n = 0, 1, \ldots , I^n \; ; \; n = A, B, \ldots , N$$ 6

the quasi-decimal expression Q of equation 15 is defined by the sum Q_t:

$$Q_t = \Sigma_{i^A, \ldots , i^n, \ldots , i^N} R_{i^A \ldots i^n \ldots i^N}$$ 10

$$R_{i^A \ldots i^n \ldots i^N} = A_{i^A} \ldots n_{i^n} \ldots N_{i^N}$$ 11

over the values satisfying

$$i^A + \ldots + i^n + \ldots + i^N = t ,$$ 12

$$t = 0, 1, \ldots , T ;$$ 13

$$T = I^A + \ldots + I^n + \ldots + I^N$$ 14

1. If any N-dimensional affine frame:

$$\vec{a}_1, \vec{a}_2, \ldots , \vec{a}_n, \ldots , \vec{a}_N$$ a-1

is constructed, either electrically or in any other manner and

2. the value

$$n_{i^n} \quad (n = A, B, \ldots, N \,;\ i^n = 0, 1, \ldots, I^n)$$

is entered at all points

$$(x^1, \ldots, x^{n-1}, x^{n+1}, \ldots, x^N) = (i^A, \ldots, i^{n-1}, i^{n+1}, \ldots, i^N) \qquad 18$$

on the hyperplane

$$x^n = i^n , \qquad 19$$

3. then by adding the products at first on the diagonal hyperplane defined by

a-4, a-7: $$x_n x^n = t, \quad w_n = 1 \quad (n = 1, 2, \ldots, N)$$

namely,

a-9: $$x^1 + x^2 + \ldots + x^n + \ldots + x^N = t \qquad 20$$

$$t = 0, 1, \ldots, T \qquad 20$$

14: $$T = I^A + \ldots + I^n + \ldots + I^N \qquad 20$$

Q_t is obtained and by adding these Q_t with carry over, the product P is obtained. Hence the rules of multiplication are:

1. Construct N-dimensional affine frame, where N is equal to the number of the factors A, B, ... , N ;
2. Realize n_{i^n} $(i^n = 0, 1, \ldots, I^n \,;\ n = A, B, \ldots, N)$ sheets of hyperplanes in parallel to the base covector $\underrightarrow{a}^n$ $(n = 1, 2, \ldots, N)$ at

$$x^n = i^n \quad (i^n = 0, 1, \ldots, I^n) \,; \qquad 19.6$$

3. Q_t is obtained by adding the number of intersections of these sets of hyperplanes on the diagonal hyperplanes of equation 19;

4. The product P is obtained by adding these $Q_t 10^t$ with carry over.

If the requested maximum number of digits is D for all:

$$I^n + 1 = I + 1 = D \qquad 8$$

$$T = NI = N(D - 1) \qquad 24$$

by installing

$$M_N = D^N$$

elements of multiplying N numbers, all M multiplication of

$$R_{i^A \cdots i^n \cdots i^N} = A_{i^A} \cdots n_{i^n} \cdots N_{i^N}; \; i^n = 0, 1, \ldots, I$$

from

$$R_{00 \ldots 0} = A_0 \cdots n_0 \cdots N_0$$

to

$$R_{II \ldots I} = A_I \cdots n_I \cdots N_I$$

can be simultaneously done spending the same period to the above single multiplication.

For the addition of these products to Q_t of each t, instead of repeating the addition of two numbers if the combinational circuits of the so-called symmetric Boolean functions [C-SCLD236, HiG-LODEC 83, N-rc, Sh-sarsc] of Figure 10 are used, Q_t can be obtained in a single step independent of the number of its digits. Different t can be added simultaneously.

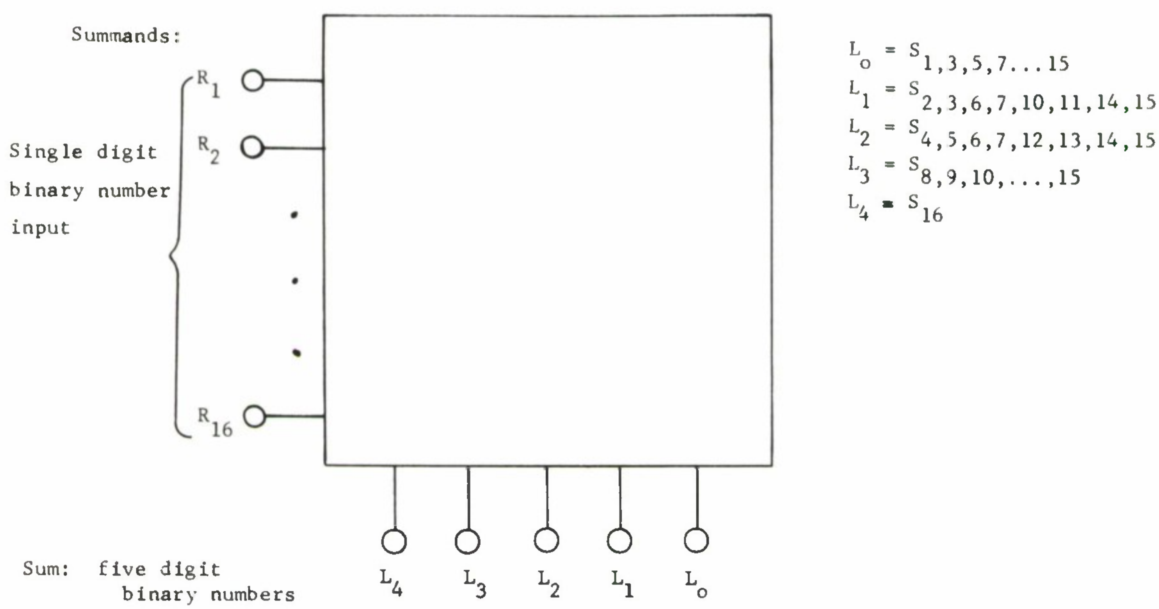


Figure 10 Combinatorial Adder

When the number of the summands are many, e. g. , 1024, by splitting these into 8 groups and then further splitting each group again into 8 subgroups, the addition is reduced to that of 16 numbers and two further addition of eight numbers. By such multiple-step splitting, the combinational circuits of the type of Figure 10 generally can be kept to a reasonable size.

Parallel operation of the addition in these 8 x 8 = 64 subgroups reduces the computing time. Series operation saves the adding components at some increase in computing time. Because the purpose of this paper is to describe the basic ideas, only a few remarks will be given on the practical realization.

At first, the error correcting and detecting facilities can be introduced by redundancy. In Figures 3 and 4, if nine conductors are installed instead of the bars, and magnetic cores are linked at all cross points, this can form a core matrix and the multiplication can be done in a single time unit by impressing the pulses simultaneously on A_i and B_j. The conductors for each decimal digit can be reduced to five as explained after 2: equation 9/2. The decimal value of each product A_iB_j can be obtained (without delay caused by the carrying process) by the use of the combinatorial circuits of the type shown in Figure 10 where each core of 9 x 9 = 81 or 5 x 5 = 25 forms the input. The addition of these to Q_t can be done again by the combinatorial circuits of type Figure 10 without carrying delay.

This can be generalized to the case N, where the magnetic cores link to the conductors of N-dimensional lattice circuit. The circuit of Figures 11, 12, and 13 is a combinatorial circuit whose output lines P_j will assume directly the values that represent the product of the input lines A and B in the binary expression, after a suitable logical delay for propagation of voltage levels through the matrix. For example:

$$P = A \times B = 10110 \times 01101 = 0100011110 \text{ ,} \qquad 25/2$$

The highest power T of the quasi-binary product Q is

$$T = I + J = 4 + 4 = 8 \text{ ;} \qquad 26/2$$

hence by possible carry, the highest power S of the product P is

$$S = T + 1 = 9 \text{ ,} \qquad 27/2$$

which decimally means

$$P = A \times B = 22 \times 18 = 286 \qquad 28/2$$

$$T = I + J = 1 + 1 = 2, \ S = T = 2 \text{ .} \qquad 29/2$$

In Figures 12 and 13 the boxes in general position have three inputs:

i) the product, namely the output of AND gate of A_i and B_j:

$$R_{ij} = A_iB_j \qquad 30/2$$

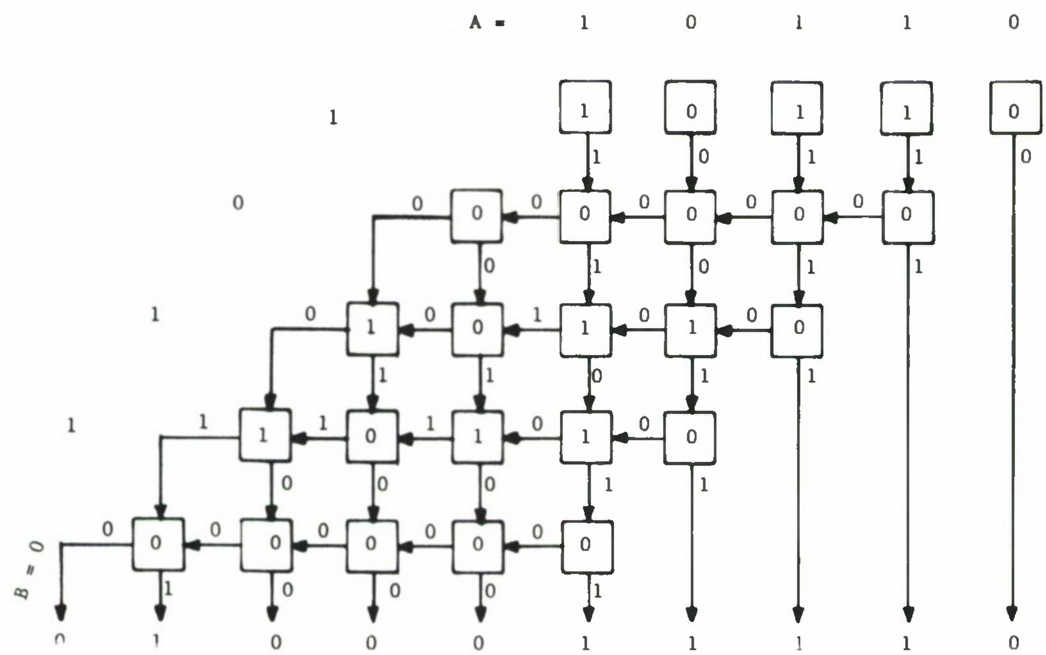


Figure 11 Multiplier of Two Binary Numbers

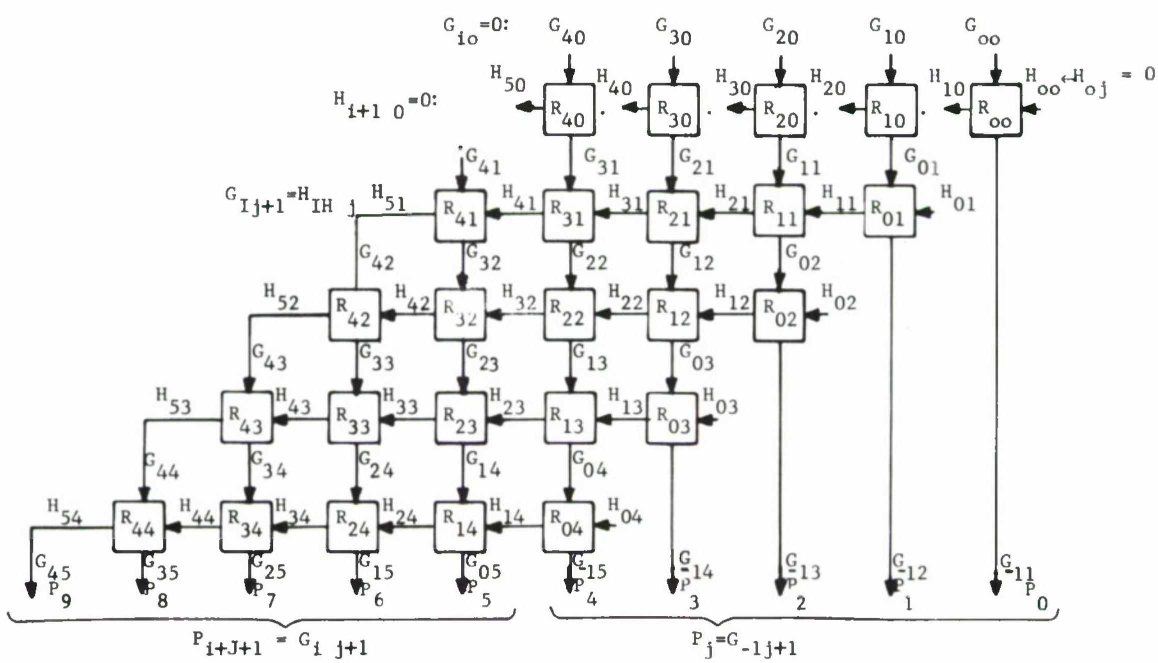


Figure 12 Circuit Construction

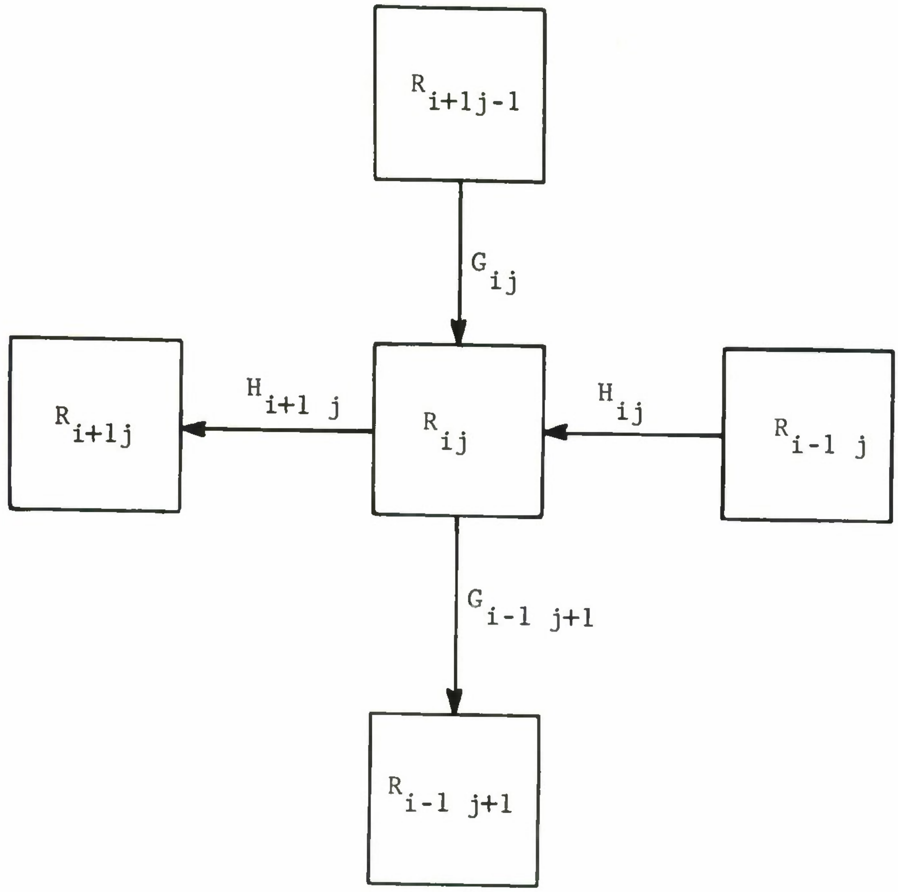


Figure 13 General Cell

ii) input from the upper box: G_{ij}

iii) input from the right box: H_{ij}

and two outputs:

i) adding $S_{1,3}$ gate to lower box:

$$G_{i-1\ j+1} = R_{ij} + G_{ij} + H_{ij} \pmod{2} \qquad 31/2$$

ii) carrying $S_{2,3}$ gate to left box:

$$H_{i+1\ j} = G_{ij}H_{ij} + G_{ij}R_{ij} + H_{ij}R_{ij} \pmod{2} \qquad 32/2$$

The first row of the first partial product has no input from above:

$$G_{i0} = 0 \tag{33/2}$$

and no output to the left:

$$H_{i+1\ 0} = 0 \tag{34/2}$$

and further no input from right; however, this condition logically results from equations 30/2, 31/2, and 32/2:

$$\begin{aligned} H_{i0} &= G_{i-1\ 0}H_{i-1\ 0} + G_{i-1\ 0}R_{i-1\ 0}\ H_{i-1\ 0}R_{i-1\ 0} \\ &= 0 \cdot 0 + 0 \cdot R_{i-1\ 0} + 0 \cdot R_{i-1\ 0} = 0\ . \end{aligned} \tag{35/2}$$

The first right skew column including the factor A_0 has no input from right

$$H_{0j} = 0\ . \tag{36/2}$$

The input of the first left skew column G_{Ij+1} comes from the adjacent skew upper right horizontal output:

$$G_{I\ j+1} = H_{I+1\ j}\ . \tag{37/2}$$

For $j = 0$, this means

$$G_{I\ 1} = H_{I+1\ 0} = 0 \tag{38/2}$$

by equation 31/2.

The first J digits $P_0, P_1, \ldots, P_J$ from the right of the product P is given by

$$P_j = G_{-1\ j+1}\ ; \tag{39/2}$$

further digits are given by

$$P_{i+J+1} = G_{iJ+1}\ . \tag{40/2}$$

If Y is the time required to form the logical product R_{ij} from A_i and B_j and Z is the time required to form $G_{i-1, j+1}$ and $H_{i+1, j}$ from the inputs R_{ij}, G_{ij} and H_{ij} through a single cell of the matrix, the longest delay occurs in the series

$$H_{11}, H_{21}, H_{31}, H_{41}, G_{32}, H_{42}, H_{42}, G_{33}, H_{43}, G_{34}, H_{44}, H_{54}, \qquad 41/2$$

hence the maximal delay U for the entire matrix is given by

$$U = Y + Z \left\{ (D-1) + 2(D-2) + 1 \right\} = Y + Z(3D-4) . \qquad 42/2$$

The case $N = 3$ can be constructed by a similar circuitry, but with some additional gates shown by the small circles at the left end of each horizontal C_k plane in Figure 9. The factors of the same radix power appear in the same vertical diagonal planes in Figure 9:

$$R_{000} = A_0B_0C_0 \; ; R_{100} = A_1B_0C_0, \; R_{010} = A_0B_1C_0 ,$$

$$R_{001} = A_0B_0C_1, \ldots , R_{444} = A_4B_4C_4 .$$

Because

$$I = J = K = 4$$

in this example, all of the A, B, and C are smaller than 2^5; hence the product P is smaller than $(2^5)^3 = 2^{15}$. Therefore, the maximal value of S in P is 14:

$$S \leq 3(I + 1) - 1 = 14 .$$

SECTION V

CONCLUSION

The above process can be summarized as follows:

1. Simultaneous expression of each numeral in all places where-ever it is used,

2. Mutual separation of the products of every pair or set of digits by removing the carryover process into the final adding step,

3. Utilization of the simple "diagonal" location of all the products of pairs or sets of digits of the same radix power, and

4. Simultaneous partial addition of the products to the quasi-term Q_t by the combinational circuits instead of the series operation of repeating the shift-and-add with carry operations of two partial products.

By assembling these geometrized arithmetic devices, any polynomial or Taylor series of finite terms can be realized in hardware, which will facilitate and expedite the numerical calculation of roots of algebraic equations.

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APPENDIX A

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DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) The MITRE Corporation Bedford, Massachusetts	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP N/A	
3. REPORT TITLE Bar-Multiplier and its Generalization		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) N/A		
5. AUTHOR(S) (Last name, first name, initial) Okada, Satio		
6. REPORT DATE December 1965	7a. TOTAL NO. OF PAGES 38	7b. NO. OF REFS 26
8a. CONTRACT OR GRANT NO. AF19(628)-5165 b. PROJECT NO. 7070 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) ESD-TDR-64-172 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) TM-04026	
10. AVAILABILITY/LIMITATION NOTICES Qualified requestors may obtain from DDC DDC release to CFSTI authorized.		
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13. ABSTRACT

A new and simple device, similar to an abacus, was invented by a Chinese mathematician; it is used for multiplication, division, and extracting square roots. The underlying idea is abstracted and generalized for possible application to contemporary computers in order to reduce computing time.

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